L14 Feb 10 Infinite Product

Tuesday, February 3, 2015 9:20 PM

Finite product $\prod_{k=1}^{n} X_k = X_1 \times X_2 \times \cdots \times X_n$

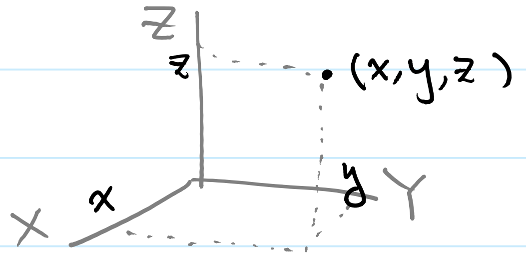
$$\mathcal{S} = \{ X_1 \times \cdots \times U_j \times \cdots \times X_n : U_j \in \mathcal{T}_j, \ j = 1, \cdots, n \}$$

$$\mathcal{B} = \{ \prod_{k=1}^{n} U_k : U_k \in \mathcal{T}_k \}$$

Infinite product $P = \prod_{\alpha \in I} X_\alpha$

Qu. How to easily give $\mathcal{S}$ analogously?

An element $(x, y, z) \in X \times Y \times Z$



Visualized Geometrically

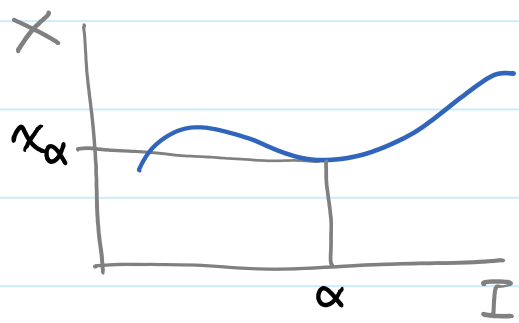
An element $x \in P = \prod_{\alpha \in I} X_\alpha$

$$x : I \longrightarrow \bigcup_{\alpha \in I} X_\alpha \qquad x(\alpha) \overset{\text{denote}}{=\!=} x_\alpha \in X_\alpha$$

e.g. $I = \{1, 2, \cdots, n\} \xrightarrow{x} (x_1, x_2, \cdots, x_n)$

In particular, when $X_\alpha = X$ for all α

$$\prod_{\alpha \in I} X_\alpha = X^I = \{ \text{mappings } I \longrightarrow X \}$$



Similar to $(x_1, x_2, \cdots, x_n) \xmapsto{\pi_k} x_k$,

We also have the projection mapping

$$\pi_\alpha : P \longrightarrow X_\alpha : x \longmapsto x_\alpha$$

With such notation,

$$X \times V = \pi_2^{-1}(V), \quad U \times Y = \pi_1^{-1}(U)$$

Naturally, in general

$$\mathcal{S} = \bigcup_{\alpha \in I} \{ \pi_\alpha^{-1}(U) : U \in \mathcal{J}_\alpha \}$$

However, if I is infinite, after finite $\cap$,

$$\mathcal{B} \neq \{ \prod_{\alpha \in I} U_\alpha : U_\alpha \in \mathcal{J}_\alpha \}$$

Example $I = \mathbb{N}$, an element of $\mathcal{B}$ is

$$X_1 \times \cdots \times U_{k_1} \times \cdots X_j \times \cdots \times U_{k_2} \times \cdots X_l \times \cdots U_{k_n} \times \cdots X_m \cdots$$

only finitely many $\subsetneq X_\alpha$

Example. Consider $I = \mathbb{N}$ and all $X_\alpha = \{0, 1\}$

Then $\prod_{\alpha \in I} X_\alpha = \{0,1\}^{\mathbb{N}}$, sequences of $0, 1$

Let $\bar{0} = (0, 0, 0, \cdots, \cdots)$, all entries $= 0$, Then

$B \in \mathcal{B}$ with $\bar{0} \in B$ is of the form

$$\{0,1\} \times \{0,1\} \times \cdots \times \{0\} \times \cdots \times \{0,1\} \times \cdots \{0\} \times \cdots \times \{0,1\} \times \cdots$$

finitely many such $\{0\}$

$\{ \prod_{\alpha \in \mathbb{N}} U_\alpha \}$ contains $\{0\} \times \{0\} \times \cdots \times \{0\} \times \cdots \times \cdots$ all $\{0\}$

which produces the discrete topology

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Natural expectation, each

$$\pi_\beta : \prod_{\alpha\in I} X_\alpha \longrightarrow X_\beta \quad \text{is continuous}$$

$$\uparrow$$

$$\underbrace{\text{Discrete} \supset \cdots \supset \mathcal{T}_\Pi}_{\text{Continuous}} \supset \cdots \supset \text{Indiscrete}$$

For this reason, $\pi_\beta^{-1}(U)$, $U \in \mathcal{T}_\beta$ must $\in \mathcal{T}_\Pi$

$\therefore$ take $\mathcal{S} = \bigcup_{\alpha\in I} \{ \pi_\alpha^{-1}(U) : U \in \mathcal{T}_\alpha \}$

It generate the minimal topology that makes all $\pi_\beta : \prod_{\alpha\in I} X_\alpha \longrightarrow X_\beta$ continuous

Qu. Is $(x,y,z) \longmapsto (x+y+z,\ xyz)$ continuous?

How do you check it?

Theorem Let W be any topological space.

A mapping $f : W \longrightarrow \prod_{\alpha\in I} X_\alpha$ is continuous

$\Leftrightarrow \forall \beta \in I \quad \pi_\beta \circ f : W \longrightarrow X_\beta$ is continuous

"$\Rightarrow$" is trivial, by each π_β and f are continuous

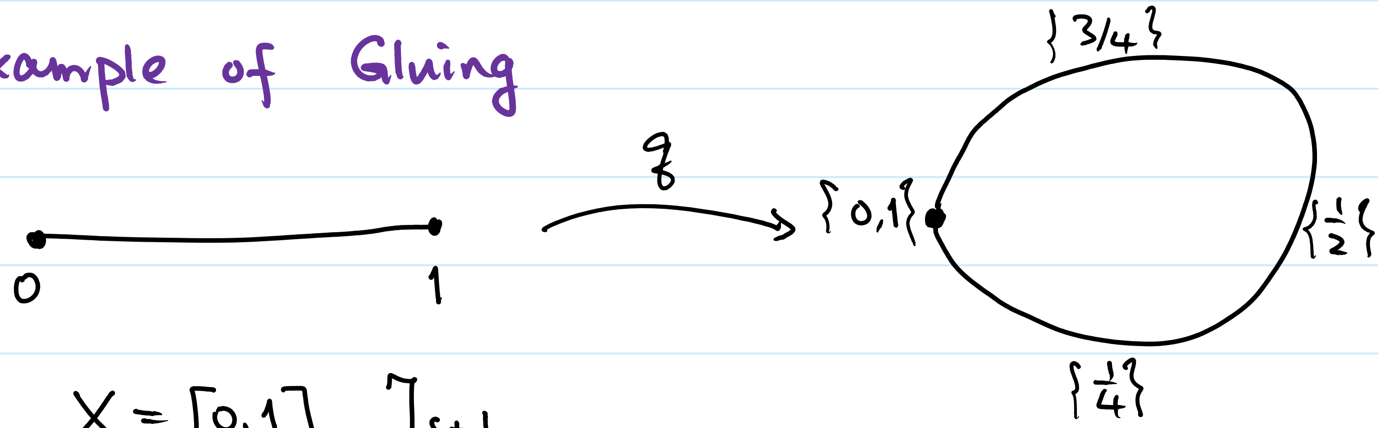
"$\Leftarrow$" Need $\forall B \in \mathcal{B}$, $f^{-1}(B)$ is open in W.

$$B = \bigcap_{k=1}^{n} \pi_{\alpha_k}^{-1}(U_k)$$

$$f^{-1}(B) = \bigcap_{k=1}^{n} f^{-1}\pi_{\alpha_k}^{-1}(U_k) = \bigcap_{k=1}^{n} (\pi_{\alpha_k}\circ f)^{-1}(U_k)$$

Exercise $\mathcal{T}_\Pi$ is also the maximal topology on $\prod X_\alpha$ to make this theorem true

Example of Gluing



$X = [0,1]$, $\mathcal{T}_{std}$

As a set, we can see the "circle" as

$X/\sim$ where $\sim$ is an equiv. relation.

For $s, t \in [0,1]$, $s \sim t$ if $|s-t| = 0, 1$

$s = t$ or $s=0, t=1$; $s=1, t=0$

In such a case,

$$X/\sim = \{ \underbrace{\{0,1\}}_{[0]=[1]}, \underbrace{\{x\}}_{[x]}, 0 < x < 1 \}$$

Qu. How to put a topology on $X/\sim$?

The relation $\sim$ is equivalent to

$$q : X \longrightarrow X/\sim \; : \; x \longmapsto [x]$$

Natural expectation

$$(X, \mathcal{T}_X) \xrightarrow{q} (X/\sim, ?) \quad \text{continuous}$$

$$\mathcal{T}_q = \{ V \subset X/\sim \; : \; q^{-1}(V) \in \mathcal{T}_X \}$$

Exercise: Verify that it is a topology.